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COUNTABLE STRUCTURES WITH A FIXED GROUP OF AUTOMORPHISMS

BY

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ABSTRACT

We prove that, given a countable group G, the set of countable structures (for a suitable language L) $\mathcal{U}_G$ whose automorphism group is isomorphic to G is a complete coanalytic set and if $G \not\simeq H$ then $\mathcal{U}_G$ is Borel inseparable from $\mathcal{U}_H$. We give also a model theoretic interpretation of this result. We prove, in contrast, that the set of countable structures for L whose automorphism group is isomorphic to $\mathbb{Z}_p^{\mathbb{N}}$, p a prime number, is $\mathbf{\Pi}_1^1\&\mathbf{\Sigma}_1^1$-complete.

Introduction

This paper is devoted to the study of those classes of countable structures (for a given countable language L) characterised by sharing a given group of automorphisms.

The set X_L of (codes of) structures for L with universe $\mathbb{N}$ is a Polish space. For G a group let $\mathcal{U}_G = \{x \in X_L \mid \mathrm{Aut}(\mathcal{A}_x) \simeq G\}$ be the set of countable structures for L whose automorphism group is isomorphic to G.

Let us consider first the case when G is countable. If the language L is very simple, $\mathcal{U}_G$ can be very simple as well. For example, if L is empty or it consists of one unary relation symbol, then every $\mathcal{U}_G$ is empty. On the other hand we prove that, as soon as the language becomes reasonably rich, the sets $\mathcal{U}_G$ are quite complicated.

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THEOREM: *Let L be a language containing infinitely many unary function symbols or a function or relation symbol of arity at least 2. Then, for every countable group G, the set $\mathcal{U}_G$ is $\mathbf{\Pi}^1_1$-complete in X_L.*

This result cannot be extended to uncountable groups. We see in fact that the following holds.

THEOREM: *Let L be as above. Then, for any prime number p, $\mathcal{U}_{\mathbb{Z}_p^{\mathbb{N}}}$ is $\mathbf{\Pi}^1_1\&\mathbf{\Sigma}^1_1$-complete, where $\mathbf{\Pi}^1_1\&\mathbf{\Sigma}^1_1$ is the class of all intersections of a coanalytic and an analytic set.*

The first theorem stated gives us a family of pairwise disjoint complete coanalytic sets indexed by countable groups up to isomorphism. This family turns out to be very entangled. More precisely, we have the following result.

THEOREM: *Let L be as above. If G and H are countable groups, with $G \not\simeq H$, then $\mathcal{U}_G$ and $\mathcal{U}_H$ are Borel inseparable.*

This theorem can be rephrased by saying that, if B is a Borel subset of X_L containing $\mathcal{U}_G$, then for every other countable group H there is an element of $\mathcal{U}_H$ which belongs to B. This overspill property can be exploited, using the correspondence between invariant Borel subsets of X_L and $L_{\omega_1\omega}$-sentences, to obtain the following model theoretic interpretation of the preceding result.

THEOREM: *Let L be as above. Let G be a countable group and suppose σ is an $L_{\omega_1\omega}$-sentence satisfied by all countable structures for L whose automorphism group is isomorphic to G. Then for every countable group H there is a countable structure for L whose automorphism group is isomorphic to H satisfying σ.*

If $L = \{R\}$, R a binary relation symbol, the preceding theorems still hold when we confine our attention to the class of countable graphs.

To obtain these results a concept which turns out to be a very powerful tool is that of a *group tree*. These are trees such that every level carries the structure of a group; they have long been used in mathematical logic for various purposes: see, for instance, [Sh76], [Ma81], [La85], [Mo93], [So95] (the terminology *group tree* is taken from [Ma81] and [So95]).

In this paper we develop a construction similar to the one used in [Ma81, appendix] and [Mo93] but in a more general context (we shall recover that one later dealing with a special case); we shall assign, in a Borel way, to each descriptive tree T a group tree S^G_T, depending on a fixed countable group G. Each level $\mathrm{Lev}_n(S^G_T)$ will be a group isomorphic to a free sum of copies of G, the number of summands — finite or countably infinite — being equal to the cardinality of $\mathrm{Lev}_n(T)$.

In the spirit of this paper the work of [Mo93] is particularly interesting. Indeed, while that paper deals with recursive model theory, several arguments work in a classical descriptive set theoretic context too.

So, if L is a countable language as above and X_L is the Polish space of (codes for) countable structures for L, we deduce from there that the set of non-rigid structures is complete analytic (a special case of the first theorem stated above) and we can also obtain a proof in ZFC that the isomorphism relation in X_L is complete analytic using just methods of classical descriptive set theory. For a proof of this fact using effective descriptive set theory see [FS89]; for a proof using classical descriptive set theory in ZFC+$\mathbf{\Sigma}^1_1$-determinacy see [Ke95, 27.D]. Another classical type proof in ZFC is due to R. Dougherty.

We begin stating some basic properties of the structures we shall be interested in.

In the second section we describe the construction of our group trees S^G_T and prove the main results; it will be more convenient to prove them in a slightly different order from the one they were stated above.

In the last section we deal with the uncountable case, working with the groups $\mathbb{Z}_p^{\mathbb{N}}$, p a prime number.

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1. Some algebraic preliminaries

Let S be a semigroup for which some mapping $|\ |\colon S \longrightarrow \mathbb{N}$ is defined and which contains a family $\{e_i\}_{i\in\mathbb{N}}$ of elements and such that

(1) $\forall m \in \mathbb{N}\ |e_m| = m$;
(2) $\forall m, n \in \mathbb{N}\ e_m e_n = e_{\min(m,n)}$;
(3) $\forall x, y \in S\ |xy| = \min(|x|, |y|)$;
(4) $\forall x \in S\ \forall m \in \mathbb{N}\ (x e_m = x \iff |x| \le m)$;
(5) $\forall x \in S\ \forall m \in \mathbb{N}\ x e_m = e_m x$.

The function $|\ |$ will also be called *rank* in the sequel.

For $x, y \in S$ define $x \preceq y \iff x = e_{|x|} y$ and $x \prec y \iff x \preceq y \wedge x \neq y$.

LEMMA 1: *$\forall x, y \in S$ ($x \preceq y \iff \exists m \in \mathbb{N}\ x = e_m y$).*

Proof: The forward implication is immediate. So assume $\exists m \in \mathbb{N}\ x = e_m y$. By (3) and (1) $|x| \leq m$ so, using (4), (5) and (2), $x = e_{|x|} e_m y = e_{|x|} y$.

LEMMA 2: *$\preceq$ is a partial order.*

Proof: By (5) and (4), $\forall x \in S\ e_{|x|} x = x e_{|x|} = x$ whence $x \preceq x$, proving reflexivity. Assume now $x \preceq y \preceq z$, that is $x = e_{|x|} y \wedge y = e_{|y|} z$; this implies $x = e_{\min(|x|,|y|)} z$ and $x \preceq z$, proving transitivity. Finally assume $x \preceq y \preceq x$, which means $x = e_{|x|} y \wedge y = e_{|y|} x$. Suppose $|x| \leq |y|$ (the case $|y| \leq |x|$ is symmetric). We have thus $y = e_{\min(|x|,|y|)} y = e_{|x|} e_{|y|} x = e_{|x|} x = x$ and antisymmetry is proved.

LEMMA 3: *$\forall x, y \in S$ ($x \prec y \Rightarrow |x| < |y|$).*

Proof: Let $x \prec y$. Then $x = e_{|x|} y$. The relation $|y| \leq |x|$ is impossible, since (4), (5) and (2) would then imply $x = e_{|x|} e_{|y|} y = e_{|y|} y = y$.

LEMMA 4: *$\forall x, y, z, t \in S$ ($x \preceq y \wedge z \preceq t \Rightarrow xz \preceq yt$).*

Proof: We have $x = e_{|x|} y$ and $z = e_{|z|} t$. So $xz = e_{\min(|x|,|z|)} yt \preceq yt$.

We turn now to a model $P(S) = (S, (f_\alpha)_{\alpha \in S})$, where, for $\alpha \in S$, f_α is the unary function defined by $\forall x \in S\ f_\alpha(x) = \alpha x$.

LEMMA 5: *Let φ: $P(S) \longrightarrow P(S)$ be an automorphism. Then $\forall x, y \in S$ ($x \preceq y \iff \varphi(x) \preceq \varphi(y)$).*

Proof: We have

$$\begin{aligned} x \preceq y &\iff \exists m \in \mathbb{N}\ x = e_m y \iff \\ &\iff \exists m \in \mathbb{N}\ \varphi(x) = \varphi(e_m y) = e_m \varphi(y) \iff \\ &\iff \varphi(x) \preceq \varphi(y). \end{aligned}$$

LEMMA 6: *Let φ: $P(S) \longrightarrow P(S)$ be an automorphism. Then there exists a strictly increasing chain $\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_n \prec \cdots$ of elements of S such that $\forall x \in S\ \forall m \geq |x|\ \varphi(x) = x\alpha_m$.*

Proof: Assume $m \geq |x|$. By (4), $\varphi(x) = \varphi(xe_m) = x\varphi(e_m)$. Since (1) and (2) imply $\forall h, k \in \mathbb{N}$ ($e_h \prec e_k \iff h < k$), using Lemma 5 we get $\varphi(e_0) \prec \varphi(e_1) \prec \cdots \prec \varphi(e_n) \prec \cdots$ It is enough to put $\forall h \in \mathbb{N}\ \alpha_h = \varphi(e_h)$.

LEMMA 7: *Let φ: $P(S) \longrightarrow P(S)$ be an automorphism. Then $\forall x \in S$ $|\varphi(x)| = |x|$.*

Proof: By Lemma 6, let $\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_n \prec \cdots$ be a strictly increasing chain of elements of S such that $\forall x \in S\ \forall m \geq |x|\ \varphi(x) = x\alpha_m$. By Lemma 3 we have $|\alpha_0| < |\alpha_1| < \cdots < |\alpha_n| < \cdots$ and this implies $\forall n \in \mathbb{N}\ |\alpha_n| \geq n$. So, for $m \geq |x|$, we have $|\alpha_m| \geq m \geq |x|$ and thus $|\varphi(x)| = |x\alpha_m| = |x|$.

LEMMA 8: *Let $\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_n \prec \cdots$ be a strictly increasing chain in S. Then $\forall x \in S\ \forall n \geq |x|\ x\alpha_n = x\alpha_{|x|}$ and, defining $\forall x \in S\ \varphi(x) = x\alpha_{|x|}$, φ: $P(S) \longrightarrow P(S)$ is a morphism.*

Proof: We have, as before, $\forall n \in \mathbb{N}\ |\alpha_n| \geq n$. Let $x \in S$ and let $k \in \mathbb{N}$ with $|x| < k$. Thus $|\alpha_k| > |\alpha_{|x|}| \geq |x|$. Since $\alpha_{|x|} \prec \alpha_k$ we have $\alpha_{|x|} = e_{|\alpha_{|x|}|}\alpha_k$ whence $x\alpha_{|x|} = xe_{|\alpha_{|x|}|}\alpha_k = x\alpha_k$.

We prove now that the function φ: $P(S) \longrightarrow P(S)$ defined by $\forall x \in S\ \varphi(x) = x\alpha_{|x|}$ is a morphism. Assume $\beta, x \in S$. Recalling that $|\beta x| \leq |x|$, by the first part of the proof we have $\varphi(\beta x) = \beta x\alpha_{|\beta x|} = \beta x\alpha_{|x|} = \beta\varphi(x)$.

Remark: $(S, \preceq)$ is a forest, where $\forall n \in \mathbb{N}\ \mathrm{Lev}_n(S) = \{x \in S \mid |x| = n\}$. Indeed, for $x \in \mathrm{Lev}_n(S)$ and $m < n$, x has exactly one predecessor in $\mathrm{Lev}_m(S)$, namely $e_m x$.

So, adding a common root, we can view S as a tree in the descriptive set theoretic meaning. If S is countable we can thus identify $[S]$ with a closed subset of the Baire space $\mathbb{N}^{\mathbb{N}}$. By Lemmas 5 and 7, every automorphism of $P(S)$ induces an isometry of $([S], d)$ where, for $\xi, \eta \in [S]$, $d(\xi, \eta) = 2^{-n-1}$, if $\xi \neq \eta$ and n is the first level where ξ and η differ.

2. The main construction

Let now G be a countable group. To each $T \in \mathrm{Tr}$ we associate a semigroup S_T^G (denoted S_T in the sequel, if no ambiguity arises) via generating elements and defining relations. Noting that, for each $g \in G$, $T \times \{g\}$ is a tree isomorphic to T, let $I = T \times G$ and consider a set of new elements $E = \{e_k\}_{k\in\mathbb{N}}$, where $\forall k, k' \in \mathbb{N}\ (k \neq k' \Rightarrow e_k \neq e_{k'})$. Let $I \cup E$ be the set of generators for S_T.

The relations between the generators of the semigroup are the following:

(a) $\forall \gamma, \gamma' \in I\ (\mathrm{length}(\gamma) < \mathrm{length}(\gamma') \Rightarrow \gamma\gamma' = \gamma(\gamma' \restriction \mathrm{length}(\gamma)) \wedge \gamma'\gamma = (\gamma' \restriction \mathrm{length}(\gamma))\gamma)$;

(b) $\forall \alpha \in T\ \forall g, g' \in G\ (\alpha, g)(\alpha, g') = (\alpha, gg')$;

(c) $\forall m, n \in \mathbb{N}\ e_m e_n = e_{\min(m,n)}$;

(d) $\forall\gamma \in I\ \forall n \in \mathbb{N}\ e_n\gamma = \gamma e_n = \gamma \restriction \min(\text{length}(\gamma), n)$;
(e) $\forall\alpha \in T\ (\alpha, 1_G) = e_{\text{length}(\alpha)}$.

Remark: The last condition on the generators kills immediately the copy $T \times \{1_G\}$ of T, forcing each element in $\text{Lev}_n(T)$, for $n \in \mathbb{N}$, to be equal to e_n. So if we restrict ourselves to the G_δ set $\text{Tr}^* = \{T \in \text{Tr} \mid \forall n \in \mathbb{N}\ \exists\alpha \in \mathbb{N}^n\ \alpha \in T\}$, we can avoid the use of the set $E = \{e_k\}_{k\in\mathbb{N}}$, deleting (e) and substituting (c) and (d) with

(c′) $\forall m, n \in \mathbb{N}\ (m \leq n \Rightarrow \forall\alpha \in \text{Lev}_m(T)\ \forall\beta \in \text{Lev}_n(T)\ (\alpha, 1_G)(\beta, 1_G) = (\alpha, 1_G))$;
(d′) $\forall\gamma \in I\ \forall n \in \mathbb{N}\ \forall\alpha \in \text{Lev}_n(T)\ (\alpha, 1_G)\gamma = \gamma(\alpha, 1_G) = \gamma \restriction \min(\text{length}(\gamma), n)$

respectively. Notice that the relation (c′) identifies all the elements in the same level of $T \times \{1_G\}$.

This is an equivalent construction, allowing us to get the same results as in the sequel.

Define now the rank $|\ |\colon S_T \longrightarrow \mathbb{N}$. For generators let $\forall k \in \mathbb{N}\ \forall\gamma \in I\ (|e_k| = k \wedge |\gamma| = \text{length}(\gamma))$. If $t = \beta_1 \cdots \beta_r \in S_T$, with $\beta_j \in I \cup E$, define $|t| = \min(|\beta_1|, \ldots, |\beta_r|)$. Notice that this is well defined, being independent of the choice of the word representing t, since the use of any one of the above relations does not change the minimum of the ranks of the generators involved.

Definition: Let $t = \beta_1 \cdots \beta_r \in S_T$, with $\beta_j \in I \cup E$. The expression $\beta_1 \cdots \beta_r$ is a **canonical form** for t if

- $|\beta_1| = \cdots = |\beta_r| = |t|$,
- no substring of the forms $(\alpha, 1_G)$ and $(\alpha, g)(\alpha, g')$ occurs in $\beta_1 \cdots \beta_r$,
- either no element of $\{e_k\}_{k\in\mathbb{N}}$ occurs in $\beta_1 \cdots \beta_r$ or $r = 1 \wedge \exists k \in \mathbb{N}\ \beta_1 = e_k$.

LEMMA 9: *Every element of S_T has exactly one canonical form.*

Proof: Let $t = \beta_1 \cdots \beta_r \in S_T$, with $\beta_j \in I \cup E$. If $\beta_1 \cdots \beta_r$ does not contain elements of I, then $t = \beta_1 \cdots \beta_r = e_{b_1} \cdots e_{b_r} = e_{\min(b_1,\ldots,b_r)}$. Otherwise let $\delta_1 \cdots \delta_p$ be the expression obtained from $\beta_1 \cdots \beta_r$ after deleting all occurrences of elements from $(T \times \{1_G\}) \cup \{e_k\}_{k\in\mathbb{N}}$ and restricting the others to $|t|$. Then keep substituting strings of elements of the form $(\alpha, g)(\alpha, g')$, for $\alpha \in T$ and $g, g' \in G$, with (α, gg') and erasing any $(\alpha, 1_G)$ appearing so that the process must eventually stop. If the final sequence is not empty, that is a canonical form for t; if it is empty, then $t = e_{|t|}$. Now we prove uniqueness. Assume $\beta_1 \cdots \beta_r = \delta_1 \cdots \delta_p$ where the expressions occurring on both sides of equality are canonical forms for $t \in S_T$. If $\exists b \in \mathbb{N}\ t = e_b$ then $r = p = 1$ and $\beta_1 = \delta_1 = e_b$.

Otherwise let $\beta_i = (\alpha_i, g_i)$ and $\delta_j = (\alpha'_j, g'_j)$ with $g_i \neq 1_G \neq g'_j$, $\alpha_i \neq \alpha_{i+1}$, $\alpha'_j \neq \alpha'_{j+1}$, $\text{length}(\alpha_i) = \text{length}(\alpha'_j) = |t|$. When we operate with the relations (a)–(e) on our equality, any expression we can obtain from $\beta_1 \cdots \beta_r$ must have the form $X_1(\alpha_{11}, g_{11}) \cdots (\alpha_{1n_1}, g_{1n_1})X_2 \cdots X_r(\alpha_{r1}, g_{r1}) \cdots (\alpha_{rn_r}, g_{rn_r})X_{r+1}$ where

- $X_i = e_n$ for some $n \geq |t|$;
- $\prod_{h=1}^{n_i} g_{ih} = g_i$;
- $\alpha_i \subseteq \alpha_{ih}$;
- at least one letter in the expression has rank equal to $|t|$.

Similarly for $\delta_1 \cdots \delta_p$. In order to have a letter-by-letter equality between two such expressions we must have $r = p$ and $\beta_i = \delta_i$ for all i.

Now it is possible to prove that the semigroup S_T meets the requirements (1)–(5):

(1): $\forall m \in \mathbb{N}\ |e_m| = m$ by definition;

(2): $\forall m, n \in \mathbb{N}\ e_m e_n = e_{\min(m,n)}$ by relation (c);

(3): let $x = \gamma_1 \cdots \gamma_r$ and $y = \gamma'_1 \cdots \gamma'_p$, with $\gamma_j, \gamma'_j \in I \cup E$. Then $|xy| = \min(|\gamma_1|, \ldots, |\gamma_r|, |\gamma'_1|, \ldots, |\gamma'_p|) = \min(|x|, |y|)$;

(4): let $x = \beta_1 \cdots \beta_r$ in canonical form and $m \in \mathbb{N}$. If $\exists b \in \mathbb{N}\ x = e_b$, then $xe_m = x \iff e_b e_m = e_{\min(b,m)} = e_b \iff |x| = b \leq m$. Otherwise $xe_m = (\beta_1 \restriction \min(|x|, m)) \cdots (\beta_r \restriction \min(|x|, m)) = x \iff |x| \leq m$;

(5): let $x = \beta_1 \cdots \beta_r$ in canonical form. If $\exists b \in \mathbb{N}\ x = e_b$ we have $xe_m = e_b e_m = e_{\min(b,m)} = e_m e_b = e_m x$. Otherwise $xe_m = (\beta_1 \restriction \min(|x|, m)) \cdots (\beta_r \restriction \min(|x|, m)) = e_m x$.

Thus we can define $\preceq$ and $\prec$ in S_T. Note that $\forall \alpha, \alpha' \in T\ \forall g, g' \in G$ $((\alpha, g) \preceq (\alpha', g') \iff g = g' \wedge \alpha \subseteq \alpha')$. Indeed $(\alpha, g) \preceq (\alpha', g') \iff (\alpha, g) = e_{\text{length}(\alpha)}(\alpha', g') = (\alpha' \restriction \text{length}(\alpha), g')$.

LEMMA 10: *For each $n \in \mathbb{N}$, $\text{Lev}_n(S_T)$ is a group. It is generated by the elements of $(I \cup E) \cap \text{Lev}_n(S_T)$ with those relations of S_T involving only elements of $\text{Lev}_n(S_T)$ and it is isomorphic to a free sum of copies of G, the number of summands being equal to the cardinality of $\text{Lev}_n(T)$.*

Proof: By (4) and (5), e_n is the identity element in $\text{Lev}_n(S_T)$. Let $x = (\alpha_1, g_1) \cdots (\alpha_r, g_r) \in \text{Lev}_n(S_T) \setminus \{e_n\}$ be in canonical form. Then

$$x(\alpha_r, g_r^{-1}) \cdots (\alpha_1, g_1^{-1}) = (\alpha_r, g_r^{-1}) \cdots (\alpha_1, g_1^{-1})x = e_n.$$

For the second claim notice that, for obtaining an equality between words in $\text{Lev}_n(S_T)$, it is enough to operate with substitutions involving only elements whose rank is n. Indeed, we cannot use elements of rank less than n, since this would change the rank of the whole expression, and any relation involving elements of rank greater than n can be replaced in the substitution by the relation involving the corresponding restrictions to n.

For the last assertion note that in a word whose letters are all from $\mathrm{Lev}_n(T)$ we can only simplify strings of the form $(\alpha, g)(\alpha, g')$ with (α, gg') and erase letters of the form $(\alpha, 1_G)$, like in the free sum of copies of G, where each summand is indexed by the appropriate α.

Definition: Let $x \in S_T$. Define by cases an element $x^* \in S_T$. If $\exists b \in \mathbb{N}\ x = e_b$, put $x^* = x = e_b$; otherwise let $x = (\alpha_1, g_1) \cdots (\alpha_r, g_r)$ be in canonical form. Put $x^* = (\alpha_r, g_r^{-1}) \cdots (\alpha_1, g_1^{-1})$. Notice that the last equality gives x^* in its canonical form.

The operation $x \mapsto x^*$ associates to each $x \in S_T$ its inverse in the group $\mathrm{Lev}_{|x|}(S_T)$ (in particular $|x^*| = |x|$).

LEMMA 11: $\forall x, y \in S_T\ (x \prec y \Rightarrow x^* \prec y^*)$.

Proof: Let $x = (\alpha_1, g_1) \cdots (\alpha_r, g_r)$ and $y = (\eta_1, h_1) \cdots (\eta_p, h_p)$ in canonical forms. The hypothesis $x \prec y$ says that

$$(\alpha_1, g_1) \cdots (\alpha_r, g_r) = (\eta_1 \restriction |x|, h_1) \cdots (\eta_p \restriction |x|, h_p).$$

This means that, using the relations between generators of S_T, operating on the last equality we can obtain in a finite number of steps the same expression on both sides. As we noted above, we can restrict ourselves to use relations involving only elements in $\mathrm{Lev}_{|x|}(S_T)$. So consider the canonical forms for x^* and y^* and each time you used a relation of the form $(\delta, g)(\delta, h) = (\delta, gh)$ for verifying the equality $(\alpha_1, g_1) \cdots (\alpha_r, g_r) = (\eta_1 \restriction |x|, h_1) \cdots (\eta_p \restriction |x|, h_p)$ use now the relation $(\delta, h^{-1})(\delta, g^{-1}) = (\delta, (gh)^{-1})$ starting with the words $(\alpha_r, g_r^{-1}) \cdots (\alpha_1, g_1^{-1})$ and $(\eta_p \restriction |x|, h_p^{-1}) \cdots (\eta_1 \restriction |x|, h_1^{-1})$. This allows one to check the equality $(\alpha_r, g_r^{-1}) \cdots (\alpha_1, g_1^{-1}) = (\eta_p \restriction |x|, h_p^{-1}) \cdots (\eta_1 \restriction |x|, h_1^{-1})$, that is $x^* \prec y^*$.

The same argument shows the result also in case $\exists k \in \mathbb{N}\ x = e_k$. Finally, if $\exists k \in \mathbb{N}\ y = e_k$ then $\exists k' < k\ x = e_{k'}$ and the assertion is still true.

Definition: A **complete chain** in S_T is a strictly increasing chain $x_0 \prec x_1 \prec \cdots \prec x_n \prec \cdots$ of elements of S_T such that $\forall n \in \mathbb{N}\ |x_n| = n$.

LEMMA 12: *Let $x_0 \prec x_1 \prec \cdots \prec x_n \prec \ldots$be a complete chain in S_T. Define φ: $P(S_T) \longrightarrow P(S_T)$ by $\forall y \in S_T\ \varphi(y) = y x_{|y|}$. Then φ is an automorphism.*

Proof: By Lemma 8, φ is a morphism. So it is enough to prove the existence of an inverse for φ.

Define $\forall y \in S_T\ \psi(y) = yx^*_{|y|}$. Since, by Lemma 11, $x^*_0 \prec x^*_1 \prec \cdots \prec x^*_n \prec \cdots$ is a chain, by Lemma 8 ψ is a morphism. We have then $\forall y \in S_T\ (\varphi\psi(y) = yx^*_{|y|}x_{|x^*_{|y|}|} = yx^*_{|y|}x_{|y|} = y \wedge \psi\varphi(y) = yx_{|y|}x^*_{|x_{|y|}|} = yx_{|y|}x^*_{|y|} = y)$, whence $\psi = \varphi^{-1}$.

LEMMA 13: *Denote with C_T the set of complete chains of S_T. For $X = \{x_0 \prec x_1 \prec \cdots \prec x_n \prec \cdots\}$ and $Y = \{y_0 \prec y_1 \prec \cdots \prec y_n \prec \cdots\}$ in C_T define $XY = \{x_0y_0 \prec x_1y_1 \prec \cdots \prec x_ny_n \prec \cdots\}$. Then C_T is a group.*

Proof: First note that, by Lemma 4 and condition (3), $XY \in C_T$. Recalling that $E = \{e_0 \prec e_1 \prec \cdots \prec e_n \prec \cdots\}$ we have $\forall X \in C_T\ EX = XE = X$. Putting, for $X = \{x_0 \prec x_1 \prec \cdots \prec x_n \prec \cdots\} \in C_T$, $X^* = \{x^*_0 \prec x^*_1 \prec \cdots \prec x^*_n \prec \cdots\}$, we have $XX^* = X^*X = E$.

LEMMA 14: *Let $C_T^{op} = (C_T, *)$ be the opposite group of C_T, that is the group whose operation is defined by $\forall X, Y \in C_T\ X * Y = YX$. Then $\mathrm{Aut}(P(S_T)) \simeq C_T^{op}$.*

Proof: Let Θ: $\mathrm{Aut}(P(S_T)) \longrightarrow C_T^{op}$ be defined by putting, for $\varphi \in \mathrm{Aut}(P(S_T))$, $\Theta(\varphi) = \{\varphi(e_0) \prec \varphi(e_1) \prec \cdots \prec \varphi(e_n) \prec \cdots\}$. If $\varphi, \psi \in \mathrm{Aut}(P(S_T))$ we have

$$\begin{aligned}\Theta(\varphi\psi) &= \{\varphi\psi(e_0) \prec \varphi\psi(e_1) \prec \cdots \prec \varphi\psi(e_n) \prec \cdots\} \\ &= \{\varphi(\psi(e_0)e_0) \prec \varphi(\psi(e_1)e_1) \prec \cdots \prec \varphi(\psi(e_n)e_n) \prec \cdots\} \\ &= \{\psi(e_0)\varphi(e_0) \prec \psi(e_1)\varphi(e_1) \prec \cdots \prec \psi(e_n)\varphi(e_n) \prec \cdots\} \\ &= \Theta(\varphi) * \Theta(\psi).\end{aligned}$$

Since, by the proof of Lemma 6, every automorphism of $P(S_T)$ is determined by the values it takes on the elements of $\{e_k\}_{k\in\mathbb{N}}$, Θ is injective; by Lemma 12, Θ is surjective too.

THEOREM 1: *If $[T] = \emptyset$ then $e_0 \prec e_1 \prec \cdots \prec e_n \prec \cdots$ is the only complete chain in S_T and $\mathrm{Aut}(P(S_T)) = \{1\}$.*

Proof: The claim about complete chains is equivalent, by Lemma 14, to the assertion concerning $\mathrm{Aut}(P(S_T))$. So let $x_0 \prec x_1 \prec \cdots \prec x_n \prec \ldots$ be a complete chain in S_T different from $e_0 \prec e_1 \prec \cdots \prec e_n \prec \cdots$, towards a contradiction. Since $(S_T, \preceq)$ is a forest, we have $\exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ x_n \neq e_n$. Consider elements $x = (\alpha_1, g_1) \cdots (\alpha_r, g_r) \in S_T \setminus \{e_k\}_{k\in\mathbb{N}}$ and $y = (\eta_1, h_1) \cdots (\eta_p, h_p) \in S_T \setminus \{e_k\}_{k\in\mathbb{N}}$ in canonical forms, with $x \prec y$ or equivalently $(\alpha_1, g_1) \cdots (\alpha_r, g_r) = (\eta_1 \restriction |x|, h_1) \cdots (\eta_p \restriction |x|, h_p)$. Since, by Lemma 10, $\mathrm{Lev}_{|x|}(S_T)$ is a group, the last equality means that we can obtain the word on the left side by modifying the word on the right using the relations in the group (see [Ko80, page 178]). Since each relation in $\mathrm{Lev}_{|x|}(S_T)$ is of the form $(\alpha, 1_G) = e_{\mathrm{length}(\alpha)} = e_{|x|}$ or $(\alpha, g)(\alpha, g') = (\alpha, gg')$ we conclude that, for all $1 \leq j \leq r$, α_j must have an extension in $\{\eta_1, \ldots, \eta_p\}$. Applying this to the sequence $x_{n_0} \prec x_{n_0+1} \prec \cdots$, we get a sequence (in fact one for every letter in the canonical form of x_{n_0}) $\varepsilon_{n_0} \subset \varepsilon_{n_0+1} \subset \cdots$ in T such that $\bigcup_{n=n_0}^{\infty} \varepsilon_n$ is an infinite branch of T.

THEOREM 2: *If* $\mathrm{card}([T]) = 1$, *then* $\mathrm{Aut}(P(S_T)) \simeq G^{op}$.

Proof: By Lemma 14 it is enough to prove $G \simeq C_T$.

Let $[T] = \{\xi\}$. First we claim that, if $x_0 \prec x_1 \prec \cdots \prec x_n \prec \cdots$ is a complete chain in S_T, then it is completely determined by its first term x_0. To prove this note that, by the argument used to prove Theorem 1, $\forall n \in \mathbb{N}\ \exists g \in G\ x_n = (\xi \upharpoonright n, g)$. So let $x = (\xi \upharpoonright |x|, g_1)$ and $y = (\xi \upharpoonright |y|, g_2)$ be such that $x \prec y$, that is $(\xi \upharpoonright |x|, g_1) = (\xi \upharpoonright |x|, g_2)$. This implies $g_1 = g_2$. So $\exists g \in G\ \forall n \in \mathbb{N}\ x_n = (\xi \upharpoonright n, g)$ and this proves both the claim and the theorem.

THEOREM 3: *Let* WF *and* UB *be the set of wellfounded trees and the set of trees with exactly one infinite branch respectively. Then* WF $\times$ UB *and* UB $\times$ WF *are Borel inseparable in* Tr^2.

Proof: Let $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ be the Baire space and let $(F_0, F_1) \in (\mathbf{\Pi}_1^0(\mathcal{N} \times \mathcal{N}^2))^2$ be a universal pair for $\mathbf{\Pi}_1^0(\mathcal{N}^2)$, that is: if $\mathcal{F}_0, \mathcal{F}_1 \in \mathbf{\Pi}_1^0(\mathcal{N}^2)$ there is $x \in \mathcal{N}$ such that $\mathcal{F}_0 = (F_0)_x$ and $\mathcal{F}_1 = (F_1)_x$.

Let A and B be the $\mathbf{\Pi}_1^1$ predicates in $\mathcal{N}^2$ defined by $A(x,y) \iff \neg\exists z \in \mathcal{N}\ (x,y,z) \in F_0 \wedge \exists! z \in \mathcal{N}\ (x,y,z) \in F_1$ and $B(x,y) \iff \exists! z \in \mathcal{N}\ (x,y,z) \in F_0 \wedge \neg\exists z \in \mathcal{N}\ (x,y,z) \in F_1$.

Let T_0 be the tree of F_0 and T_1 be the tree of F_1. Then, for x and y in $\mathcal{N}$, $A(x,y) \iff T_0(x,y) \in \mathrm{WF} \wedge T_1(x,y) \in \mathrm{UB} \iff (T_0(x,y), T_1(x,y)) \in \mathrm{WF} \times \mathrm{UB}$ and $B(x,y) \iff T_0(x,y) \in \mathrm{UB} \wedge T_1(x,y) \in \mathrm{WF} \iff (T_0(x,y), T_1(x,y)) \in \mathrm{UB} \times \mathrm{WF}$. So, if WF $\times$ UB and UB $\times$ WF are Borel separable, so are A and B. So assume A and B are Borel separated by C. We shall show that $\mathcal{N}^2 \setminus C$ is universal for $\mathbf{B}(\mathcal{N})$, a contradiction.

Fix $D \in \mathbf{B}(\mathcal{N})$. Then there are closed sets $\mathcal{F}_0$, $\mathcal{F}_1$ in $\mathcal{N}^2$ such that for $y \in \mathcal{N}$,

$$y \in D \iff \exists z \in \mathcal{N}\ (y,z) \in \mathcal{F}_0 \iff \exists! z \in \mathcal{N}\ (y,z) \in \mathcal{F}_0$$

and

$$y \notin D \iff \exists z \in \mathcal{N}\ (y,z) \in \mathcal{F}_1 \iff \exists! z \in \mathcal{N}\ (y,z) \in \mathcal{F}_1.$$

Fix $x \in \mathcal{N}$ such that $\mathcal{F}_0 = (F_0)_x$ and $\mathcal{F}_1 = (F_1)_x$. Then we claim that $\forall y \in \mathcal{N}\ (y \in D \iff (x,y) \notin C)$. We have indeed $y \in D \Rightarrow \neg y \notin D \Rightarrow \neg\exists z \in \mathcal{N}\ (x,y,z) \in F_1$ and also $y \in D \Rightarrow \exists! z \in \mathcal{N}\ (x,y,z) \in F_0$, so $y \in D \Rightarrow (x,y) \in B \Rightarrow (x,y) \notin C$. Conversely, we have $y \notin D \Rightarrow \neg\exists z \in \mathcal{N}\ (x,y,z) \in F_0 \wedge \exists! z \in \mathcal{N}\ (x,y,z) \in F_1$, so $y \notin D \Rightarrow (x,y) \in A \Rightarrow (x,y) \in C$.

THEOREM 4: *Let $L = \{f_n\}_{n\in\mathbb{N}}$ be the language consisting of a countable infinity of unary function symbols and X_L be the set of (codes for) countably infinite structures for L (see* [Ke95, 16.C]*). Let G and H be countable groups. Then $\{x \in X_L \mid \mathrm{Aut}(\mathcal{A}_x) \simeq G\}$ and $\{x \in X_L \mid \mathrm{Aut}(\mathcal{A}_x) \simeq H\}$ are Borel inseparable.*

Proof: Given a countable group G, the map Φ_G: $\mathrm{Tr} \longrightarrow X_L$ constructed above, which associates to each $T \in \mathrm{Tr}$ (the code for) $P(S_T^{G^{op}})$, is Borel, by standard arguments. As proved in Theorems 1 and 2, $\forall T \in \mathrm{Tr}((T \in \mathrm{WF} \Rightarrow \mathrm{Aut}(\mathcal{A}_{\Phi_G(T)}) = \{1\}) \wedge (T \in \mathrm{UB} \Rightarrow \mathrm{Aut}(\mathcal{A}_{\Phi_G(T)}) \simeq G))$. Let Ψ: $\mathrm{Tr}^2 \longrightarrow X_L$ be defined by $\mathcal{A}_{\Psi(S,T)} = \mathcal{A}_{\Phi_G(S)} \oplus \mathcal{A}_{\Phi_H(T)}$ where, if $\mathcal{B}$ and $\mathcal{C}$ are L-structures, we define $\mathcal{B}\oplus\mathcal{C}$ as the structure whose universe is the disjoint union of the universes B and C of $\mathcal{B}$ and $\mathcal{C}$ respectively and whose operations $\{f_n^{\mathcal{B}\oplus\mathcal{C}}\}_{n\in\mathbb{N}}$ are defined by $\forall n \in \mathbb{N}\ ((f_{2n}^{\mathcal{B}\oplus\mathcal{C}} \restriction B) = f_n^{\mathcal{B}} \wedge \forall u \in C\ f_{2n}^{\mathcal{B}\oplus\mathcal{C}}(u) = u \wedge \forall v \in B\ f_{2n+1}^{\mathcal{B}\oplus\mathcal{C}}(v) = v \wedge (f_{2n+1}^{\mathcal{B}\oplus\mathcal{C}} \restriction C) = f_n^{\mathcal{C}})$.

We have thus $\mathrm{Aut}(\mathcal{A}_{\Psi(S,T)}) \simeq \mathrm{Aut}(\mathcal{A}_{\Phi_G(S)}) \times \mathrm{Aut}(\mathcal{A}_{\Phi_H(T)})$. Indeed, if φ_0 and φ_1 are automorphisms of $\mathcal{A}_{\Phi_G(S)}$ and $\mathcal{A}_{\Phi_H(T)}$ respectively, then their disjoint union $\varphi_0 \cup \varphi_1$ is an automorphism of $\mathcal{A}_{\Psi(S,T)}$.

Conversely, let ψ be an automorphism of $\mathcal{A}_{\Psi(S,T)}$; it is enough to show that every element u in the universe $|\mathcal{A}_{\Phi_G(S)}|$ of $\mathcal{A}_{\Phi_G(S)}$ is sent by ψ to an element in $|\mathcal{A}_{\Phi_G(S)}|$ and similarly for $|\mathcal{A}_{\Phi_H(T)}|$. Assume, towards a contradiction, $\psi(u) \in |\mathcal{A}_{\Phi_H(T)}|$. Then $\forall n \in \mathbb{N}\ f_n^{\mathcal{A}_{\Psi(S,T)}}\psi(u) = \psi(u)$; so this means that $\forall n \in \mathbb{N}\ f_n^{\mathcal{A}_{\Phi_G(S)}}(u) = u$. This can happen only if u is the e_0-element in $\mathcal{A}_{\Phi_G(S)}$ and $\psi(u)$ is the e_0-element in $\mathcal{A}_{\Phi_H(T)}$, since every element of the structure $\mathcal{A}_{\Phi_G(S)}$, except possibly the e_0-element, is moved by something, and the same is true for $\mathcal{A}_{\Phi_H(T)}$.

For $n \in \mathbb{N}$ let e'_n be the e_n-element in $\mathcal{A}_{\Phi_G(S)}$ and e''_n the e_n-element in $\mathcal{A}_{\Phi_H(T)}$. So $\psi(e'_1) \in |\mathcal{A}_{\Phi_G(S)}|$. Let $m \in \mathbb{N}$ be such that $f_m^{\mathcal{A}_{\Phi_G(S)}}$ is the operation corresponding to the translation by e'_0. We have $f_{2m}^{\mathcal{A}_{\Psi(S,T)}}\psi(e'_1) = \psi f_{2m}^{\mathcal{A}_{\Psi(S,T)}}(e'_1) = \psi(e'_0) = e''_0 \in |\mathcal{A}_{\Phi_H(T)}|$ but, by the definition of $f_{2m}^{\mathcal{A}_{\Psi(S,T)}}$, this would imply $\psi(e'_1) \in |\mathcal{A}_{\Phi_H(T)}|$.

A similar argument works for showing that $\psi(|\mathcal{A}_{\Phi_H(T)}|) \subseteq |\mathcal{A}_{\Phi_H(T)}|$.

Then $(S,T) \in \mathrm{UB} \times \mathrm{WF} \Rightarrow \mathrm{Aut}(\Psi(S,T)) \simeq G$ and $(S,T) \in \mathrm{WF} \times \mathrm{UB} \Rightarrow \mathrm{Aut}(\Psi(S,T)) \simeq H$. Now apply Theorem 3.

THEOREM 5: *Let $L = \{f_n\}_{n\in\mathbb{N}}$ be a language consisting of a countable infinity of unary function symbols. Let G and H be countable groups and let σ be a sentence of $L_{\omega_1\omega}$ satisfied by all countably infinite structures for L whose automorphism group is isomorphic to G. Then there is a countably infinite structure for L whose automorphism group is isomorphic to H satisfying σ.*

Proof: By [Ke95, proposition 16.7] and Theorem 4.

For G a countable group, we can investigate further the properties of the Borel assignment $T \in \mathrm{Tr} \mapsto P(S_T)$ using the following three results, which extend and complement Theorems 1 and 2.

THEOREM 6: *Let* $\mathrm{card}([T]) = m \in \mathbb{N}$ *and let G_m be the free sum of m copies of G. Then the group of automorphisms of $P(S_T)$ is isomorphic to G_m^{op}.*

Proof: Since the cases $m = 0$ and $m = 1$ have already been considered in Theorems 1 and 2, we may assume $m \geq 2$. For the sake of definiteness, for $1 \leq j \leq m$, let $G \times \{j\}$ be the j-th copy of G in the sum. By Lemma 14 it is enough to prove that $C_T \simeq G_m$. Let $[T] = \{\xi_1, \ldots, \xi_m\}$ and let $n_0 \in \mathbb{N}$ be such that $\forall \xi \in [T]\ N_{\xi\restriction n_0} \cap [T] = \{\xi\}$, where $\forall t \in \mathbb{N}^{<\omega}\ N_t = \{\zeta \in \mathcal{N} \mid t \subseteq \zeta\}$. We claim that every complete chain $x_0 \prec x_1 \prec \cdots \prec x_n \prec \cdots$ in S_T is completely determined by x_{n_0}. Indeed, by an argument similar to the one used in the proofs of Theorems 1 and 2, x_{n_0} must be of the form $x_{n_0} = (\xi_{j_1} \restriction n_0, g_1) \cdots (\xi_{j_r} \restriction n_0, g_r)$, with $g_{j_i} \neq 1_G$ and $j_i \neq j_{i+1}$ (for $r = 0$, $x_{n_0} = e_{n_0}$). All such forms are possible, since they admit the extensions $(\xi_{j_1} \restriction n, g_1) \cdots (\xi_{j_r} \restriction n, g_r)$ for $n \in \mathbb{N}$ to obtain a complete chain and they are all distinct since at level n_0 all distinct infinite branches are already isolated. So, if $x = (\xi_{i_1} \restriction |x|, h_1) \cdots (\xi_{i_p} \restriction |x|, h_p)$ and $y = (\xi_{l_1} \restriction |y|, k_1) \cdots (\xi_{l_q} \restriction |y|, k_q)$ are in canonical form, with $n_0 \leq |x| < |y|$ and $x \prec y$, then $(\xi_{i_1} \restriction |x|, h_1) \cdots (\xi_{i_p} \restriction |x|, h_p) = (\xi_{l_1} \restriction |x|, k_1) \cdots (\xi_{l_q} \restriction |x|, k_q)$; since the right hand side is in canonical form too (the infinite branches being already isolated at level $|x|$), we get $p = q$, $\xi_{i_u} = \xi_{l_u}$, $h_u = k_u$. The theorem is proved noting that the set of all possible canonical forms for x_{n_0} is in isomorphic correspondence with G_m via $(\xi_{j_1} \restriction n_0, g_1) \cdots (\xi_{j_r} \restriction n_0, g_r) \mapsto (g_1, j_1) \cdots (g_r, j_r)$.

THEOREM 7: *Let $[T]$ be infinite and assume $\exists n_0 \in \mathbb{N}\ \forall \xi \in [T]\ N_{\xi\restriction n_0} \cap [T] = \{\xi\}$ (this implies* $\mathrm{card}([T]) = \aleph_0$*). If G_ω is the free sum of a countable infinity of isomorphic copies of G, then the group of automorphisms of $P(S_T)$ is isomorphic to G_ω^{op}.*

Proof: It is enough to prove $C_T \simeq G_\omega$. For $m \in \mathbb{N}$, let $G \times \{m\}$ be the m-th copy of G in the sum G_ω and let $[T] = \{\xi_k\}_{k\in\mathbb{N}}$. By the same argument used in the proof of Theorem 6, every complete chain $x_0 \prec x_1 \prec \cdots \prec x_n \prec \cdots$ in S_T is completely determined by its term x_{n_0}. Such a term has the form $(\xi_{j_1} \restriction n_0, g_1)\cdots(\xi_{j_r} \restriction n_0, g_r)$ (for $r = 0$ this is e_{n_0}), with $g_i \neq 1_G$ and $j_i \neq j_{i+1}$. This element corresponds to $(g_1, j_1)\cdots(g_r, j_r)$ in the isomorphism with G_ω.

THEOREM 8: *Let G be nontrivial and assume $\neg\exists n_0 \in \mathbb{N}\ \forall\xi \in [T]\ N_{\xi\restriction n_0} \cap [T] = \{\xi\}$ (this happens for sure if $\mathrm{card}([T]) = 2^{\aleph_0}$ but it is also compatible with $\mathrm{card}([T]) = \aleph_0$). Then $\mathrm{card}(\mathrm{Aut}(P(S_T))) = 2^{\aleph_0}$.*

Proof: Since $\mathrm{card}(\mathrm{Aut}(P(S_T))) \leq 2^{\aleph_0}$, it is enough to find an injection C: $\mathcal{P}(\mathbb{N}) \longrightarrow C_T$.

We claim that there exist an increasing sequence of natural numbers $m_0 < m_1 < \cdots < m_n < \cdots$ and sets $\{\xi_k\}_{k\in\mathbb{N}} \subseteq [T]$ and $\{\xi'_k\}_{k\in\mathbb{N}} \subseteq [T]$, where $\forall k, k' \in \mathbb{N}$ $((k \neq k' \Rightarrow \xi_k \neq \xi_{k'} \wedge \xi'_k \neq \xi'_{k'}) \wedge \xi_k \neq \xi'_{k'})$, with the property that $\forall n \in \mathbb{N}\ (\xi_n \restriction m_n = \xi'_n \restriction m_n \wedge \xi_n(m_n) \neq \xi'_n(m_n))$. This can be justified as follows: if T has a non-isolated branch ξ, than let $\xi_0, \xi'_0, \xi_1, \xi'_1, \ldots$ be infinite branches stemming, in that order, from ξ; otherwise let ξ_0 and ξ'_0 be infinite branches splitting at some level $m_0 \in \mathbb{N}$. Let $h_0 \in \mathbb{N}$ be such that both ξ_0 and ξ'_0 are isolated at level h_0 and let ξ_1 and ξ'_1 be infinite branches splitting at some level $m_1 > h_0$ and so on.

Let $A = \{j_0, j_1, \ldots\}$ be a subset of $\mathbb{N}$, where $k < k' \Rightarrow j_k < j_{k'}$; we describe the construction of an infinite chain $C(A) = \{x_0 \prec x_1 \prec \cdots \prec x_n \prec \cdots\}$ of S_T. Let $g \in G \setminus \{1_G\}$. For $0 \leq n \leq m_{j_0}$ put $x_n = e_n$; for $m_{j_0} < n \leq m_{j_1}$ let $x_n = (\xi_{j_0} \restriction n, g)(\xi'_{j_0} \restriction n, g^{-1})$; for $m_{j_1} < n \leq m_{j_2}$ let

$$x_n = (\xi_{j_0} \restriction n, g)(\xi'_{j_0} \restriction n, g^{-1})(\xi_{j_1} \restriction n, g)(\xi'_{j_1} \restriction n, g^{-1})$$

and so on. If A is finite, when A has no elements left simply proceed extending the chain trivially (in particular $C(\emptyset) = \{e_0 \prec e_1 \prec \cdots \prec e_n \prec \cdots\}$).

LEMMA 15: *Let X and Y be Polish spaces, $B \in \mathbf{B}(X \times Y)$ and*

$$A = \{x \in X \mid 0 \neq \mathrm{card}(B_x) \leq \aleph_0\}.$$

Then $A \in \mathbf{\Pi}^1_1(X)$ and there is a function f: $A \longrightarrow Y^{\mathbb{N}}$ which is $\mathbf{\Pi}^1_1$-measurable (as in [Ke95, 36.F]) *such that $\forall x \in A\ B_x = \{f(x)(n)\}_{n\in\mathbb{N}}$.*

Proof: By [Ke95, exercise 39.23] (note that the proof is in ZFC), $A^* = \{x \in X \mid \mathrm{card}(B_x) \leq \aleph_0\}$ is coanalytic and there is a function f^*: $A^* \longrightarrow Y^{\mathbb{N}}$ which is $\mathbf{\Pi}^1_1$-measurable and $\forall x \in A^*\ B_x \subseteq \{f^*(x)(n)\}_{n\in\mathbb{N}}$.

Now $\forall x \in X$ $(x \in A \iff x \in A^* \wedge \exists n \in \mathbb{N}$ $(x, f^*(x)(n)) \in B)$, so A is coanalytic too. Let $P = \{(x,n) \in X \times \mathbb{N} \mid x \in A \wedge (x, f^*(x)(n)) \in B\}$, so that P is coanalytic and $\forall x \in X$ $(x \in A \iff \exists n \in \mathbb{N}$ $P(x,n))$. By the number uniformization property (see [Ke95, definition 22.14 and theorem 35.1]), there is $h\colon A \longrightarrow \mathbb{N}$ which is $\mathbf{\Pi}^1_1$-measurable and $\forall x \in A$ $P(x, h(x))$. For $x \in A$ and $n \in \mathbb{N}$ put now

$$f(x)(n) = \begin{cases} f^*(x)(n) & \text{if } (x, f^*(x)(n)) \in B, \\ f^*(x)(h(x)) & \text{if } (x, f^*(x)(n)) \notin B. \end{cases}$$

Then f is $\mathbf{\Pi}^1_1$-measurable and $\forall x \in A$ $B_x = \{f(x)(n)\}_{n \in \mathbb{N}}$.

THEOREM 9: *Let $\mathcal{L}$ be any countable language and G be a countable group. Then $\{x \in X_{\mathcal{L}} \mid \mathrm{Aut}(\mathcal{A}_x) \simeq G\} \in \mathbf{\Pi}^1_1(X_{\mathcal{L}})$.*

Proof: Consider the subset F of $X_{\mathcal{L}} \times S_\infty$ defined by $F(x,g) \iff g \in \mathrm{Aut}(\mathcal{A}_x)$. It is closed and moreover $\forall x \in X_{\mathcal{L}}$ $F_x = \mathrm{Aut}(\mathcal{A}_x)$. Let

$$\mathcal{M}_{\aleph_0} = \{x \in X_{\mathcal{L}} \mid \mathrm{card}(\mathrm{Aut}(\mathcal{A}_x)) \leq \aleph_0\} \in \mathbf{\Pi}^1_1(X_{\mathcal{L}}).$$

Let $f\colon \mathcal{M}_{\aleph_0} \longrightarrow S_\infty^{\mathbb{N}}$ be a $\mathbf{\Pi}^1_1$-measurable function such that $\forall x \in \mathcal{M}_{\aleph_0}$ $F_x = \{f(x)(n)\}_{n \in \mathbb{N}}$, as in Lemma 15. We have $\forall x \in X_{\mathcal{L}}$ $(\mathrm{Aut}(\mathcal{A}_x) \simeq G \iff x \in \mathcal{M}_{\aleph_0} \wedge \{f(x)(n)\}_{n \in \mathbb{N}} \simeq G)$, so it is enough to show that the second condition in the conjunction is coanalytic.

Let $L_0 = \{\cdot\}$ be a language consisting of a binary function symbol, considered as the language for groups, and let $Y_{L_0} = 2^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}^2}$ be the set of (codes of) countable structures for L_0 (the factor $2^{\mathbb{N}}$ allows to consider structures whose universe is a subset of $\mathbb{N}$, in particular finite structures). Notice that there is a Borel function $g : S_\infty^{\mathbb{N}} \longrightarrow Y_{L_0}$ such that, if $z \in S_\infty^{\mathbb{N}}$ is such that $\{z(n)\}_{n \in \mathbb{N}}$ is a subgroup of S_∞, then $\mathcal{A}_{g(z)} \simeq \{z(n)\}_{n \in \mathbb{N}}$.

Then $gf\colon \mathcal{M}_{\aleph_0} \longrightarrow Y_{L_0}$ is $\mathbf{\Pi}^1_1$-measurable and $\forall x \in \mathcal{M}_{\aleph_0}$ $\mathcal{A}_{gf(x)} \simeq \mathrm{Aut}(\mathcal{A}_x)$. Since $\{y \in Y_{L_0} \mid \mathcal{A}_y \simeq G\}$ is Borel in Y_{L_0} (being an isomorphism class) we have that $\{x \in \mathcal{M}_{\aleph_0} \mid \mathcal{A}_{gf(x)} \simeq G\} = \{x \in \mathcal{M}_{\aleph_0} \mid \{f(x)(n)\}_{n \in \mathbb{N}} \simeq G\}$ is coanalytic as required.

THEOREM 10: *Let $L = \{f_n\}_{n \in \mathbb{N}}$ be a language consisting of a countable infinity of unary function symbols and let G be a countable group. Then $\{x \in X_L \mid \mathrm{Aut}(\mathcal{A}_x) \simeq G\}$ is complete coanalytic in X_L.*

Proof: We will define a Borel function $\Gamma\colon \mathrm{Tr} \longrightarrow X_L$ such that $\forall T \in \mathrm{Tr}$ $((T \in \mathrm{WF} \Rightarrow \mathrm{Aut}(\mathcal{A}_{\Gamma(T)}) \simeq G) \wedge (T \in \mathrm{IF} \Rightarrow \mathrm{card}(\mathrm{Aut}(\mathcal{A}_{\Gamma(T)})) = 2^{\aleph_0}))$. This will be the composition $\Gamma = \Phi_G \Xi \Theta$ where $\Phi_G\colon \mathrm{Tr} \longrightarrow X_L$ assigns to T (the code for)

$P(S_T^{G^{op}})$, while Ξ: Tr $\longrightarrow$ Tr and Θ: Tr $\longrightarrow$ Tr are defined as follows. For $T \in$ Tr put $t \in \Theta(T)$ if, and only if, the sequence $(t(1), t(3), t(5), \ldots, t(\text{length}(t) - 2))$ or $(t(1), t(3), t(5), \ldots t(\text{length}(t) - 1)$ — depending on whether length(t) is odd or even — is in T (thus $\Theta(T)$ contains all sequences of length 1; the sequences (m, n) and (m, n, p) are in $\Theta(T)$ just in case $(n) \in T$ and so on). So Θ sends wellfounded trees to wellfounded trees (since $\xi \in [\Theta(T)] \Rightarrow (\xi(1), \xi(3), \xi(5), \ldots) \in [T]$) and ill founded trees in trees with $2^{\aleph_0}$ infinite branches (since $\xi \in [T] \Rightarrow (m_0, \xi(0), m_1, \xi(1), \ldots) \in [\Theta(T)]$ for every $(m_0, m_1, \ldots) \in \mathcal{N}$). For $T \in$ Tr, Ξ simply adds one infinite branch; for instance let

$$\Xi(T) = \{(t(0)+1, t(1)+1, \ldots, t(\text{length}(t)-1)+1)\}_{t \in T} \cup \{0^n\}_{n \in \mathbb{N}}.$$

Thus, if $T \in$ WF then $\Xi\Theta(T) \in$ UB and $\text{Aut}(\mathcal{A}_{\Gamma(T)}) \simeq G$; if $T \in$ IF we have $\text{card}(\Xi\Theta(T)) = \text{card}(\text{Aut}(\mathcal{A}_{\Gamma(T)})) = 2^{\aleph_0}$.

By the way, note that the function Θ witnesses that $\{T \in \text{Tr} \mid \text{card}([T]) = 2^{\aleph_0}\}$ is $\mathbf{\Sigma}_1^1$-hard and Borel inseparable from WF.

Using the construction given, for instance, in [Ho93, pages 228–229], we can associate, in a Borel way, to each element $x \in X_L$, an element $x' \in X_{L'}$, where $L' = \{R\}$ is a language consisting of one binary relation symbol, in such a way that $\forall x, y \in X_L$ $(\mathcal{A}_x \simeq \mathcal{A}_y \iff \mathcal{A}_{x'} \simeq \mathcal{A}_{y'})$ and $\forall x \in X_L \text{Aut}(\mathcal{A}_x) \simeq \text{Aut}(\mathcal{A}_{x'})$. Moreover, for each $x \in X_L$, the structure $\mathcal{A}_{x'}$ is a graph. We can thus translate our results in the language L' and get the following theorems about the class $\mathcal{G} \subseteq X_{L'}$ of (codes for) countably infinite graphs.

THEOREM 4′: *Let G and H be countable groups. Then $\{x \in \mathcal{G} \mid \text{Aut}(\mathcal{A}_x) \simeq G\}$ and $\{x \in \mathcal{G} \mid \text{Aut}(\mathcal{A}_x) \simeq H\}$ are Borel inseparable.*

THEOREM 5′: *Let G and H be countable groups and let σ be a sentence of $L'_{\omega_1\omega}$ satisfied by all countably infinite graphs whose automorphism group is isomorphic to G. Then there is a countably infinite graph whose automorphism group is isomorphic to H satisfying σ.*

THEOREM 10′: *Let G be a countable group. Then $\{x \in \mathcal{G} \mid \text{Aut}(\mathcal{A}_x) \simeq G\}$ and $\{x \in X_{L'} \mid \text{Aut}(\mathcal{A}_x) \simeq G\}$ are complete coanalytic.*

Remark: We have obtained a family of complete coanalytic sets in a Polish space which are pairwise disjoint and Borel inseparable. Notice that this is a *large* family of such sets, being indexed by the class of countable groups up to isomorphism and the class of groups being Borel complete by [Me81].

A general procedure for producing pairs of disjoint Borel inseparable coanalytic sets can be found in [Be86].

The results of Theorems 4 and 5 hold for other classes of structures, in addition to graphs, which Theorems 4′ and 5′ were stated for. Indeed, if $\mathcal{L}$ is a countable language containing a relation or function symbol of arity at least 2, we can assign, in a continuous way, to each element of $\mathcal{G}$ an element of $X_{\mathcal{L}}$, preserving the isomorphism relation and in such a way that the automorphism group of the graph is equal to the automorphism group of the structure we associate to it.

It could be interesting to investigate further which classes C of countable structures, closed under isomorphism, in a given language, satisfy the analogs of Theorems 4′ and 5′. For instance, if C has the reconstruction property (that is $\forall \mathcal{A}, \mathcal{B} \in C$ $(\mathrm{Aut}(\mathcal{A}) \simeq \mathrm{Aut}(\mathcal{B}) \Rightarrow \mathcal{A} \simeq \mathcal{B})$) this is not the case. For example, let $\mathcal{L} = \{+, f_q\}_{q\in\mathbb{Q}}$ be the language for rational vector spaces and let $C \subseteq X_{\mathcal{L}}$ be the class of countable rational vector spaces. Then C has the reconstruction property.

3. Some results about $\mathbb{Z}_p$

We cannot extend the results of Theorems 10 and 10′ to uncountable groups; we can in fact prove that, for p a prime number, $\{x \in X_L \mid \mathrm{Aut}(\mathcal{A}_x) \simeq \mathbb{Z}_p^{\mathbb{N}}\}$ is $\mathbf{\Pi}^1_1\&\mathbf{\Sigma}^1_1$-complete. For this we need to simplify the above construction for the case when our group G is commutative.

To this aim we first observe — as arose in a discussion with S. Solecki — that if we are just interested in dealing with Abelian groups we can use the following structures to obtain the results given above.

Let G be a countable Abelian group (which we deal with in additive notation) and let Q^G_T (denoted Q_T in the sequel if unambiguous) be the semigroup obtained by S^G_T adding commutativity, that is adding the relations

(f) $\forall \alpha, \alpha' \in T\ \forall g, g' \in G\ (\alpha, g)(\alpha', g') = (\alpha', g')(\alpha, g)$

in the presentation of the semigroup. We point out that this is not a particular case of the above construction for the case of G Abelian, since — even in that case — the semigroup S_T is not commutative (except when T does not contain pairs of incompatible elements or $G = \{1_G\}$).

We can view Q_T as the set of all functions f from $\bigcup_{m=0}^n \mathrm{Lev}_m(T)$ (for some $n \in \mathbb{N}$) in G with finite support, such that $\forall \alpha \in T\ \forall k \geq \mathrm{length}(\alpha) f(\alpha) = \sum_{\beta \in \mathrm{Lev}_k(T), \alpha \subseteq \beta} f(\beta)$. For $n \in \mathbb{N}$, the element e_n is the function from $\bigcup_{m=0}^n \mathrm{Lev}_m(T)$ with constant value 0_G. The product of f and f' in Q_T is thus defined first restricting them to the intersection of domains (one included in the other) and then adding pointwise the values of f and f'. Since every relation in S_T still holds in Q_T, Q_T satisfies requirements (1)–(5) and the lemmas following them, and also, with the same proofs, the stated properties of S_T, except that now, thanks to commutativity of G and Q_T, we have $C_T^{op} = C_T$, $G^{op} = G$ and we should substitute, in Theorem 6, G_m and G_m^{op} with G^m and, in Theorem 7, G_ω and G_ω^{op} with $G^{<\omega}$.

Fix now a prime number p and let $G = \mathbb{Z}_p$; we obtain thus the semigroup $Q_T^{\mathbb{Z}_p}$ and the structure $P(Q_T^{\mathbb{Z}_p})$ (for $p = 2$ this is the structure appearing in [Ma81] and [Mo93]).

Particularizing our results to this case we have the following.

THEOREM 1*: *If* $[T] = \emptyset$ *then* $e_0 \prec e_1 \prec \cdots \prec e_n \prec \cdots$ *is the only complete chain in* $Q_T^{\mathbb{Z}_p}$ *and* $\mathrm{Aut}(P(Q_T^{\mathbb{Z}_p})) = \{1\}$.

THEOREM 6*: *Let* $\mathrm{card}([T]) = m \in \mathbb{N}^*$. *Then* $\mathrm{Aut}(P(Q_T^{\mathbb{Z}_p})) \simeq \mathbb{Z}_p^m$.

THEOREM 7*: *Let* $[T]$ *be infinite and assume* $\exists n_0 \in \mathbb{N}\, \forall \xi \in [T]\; N_{\xi\restriction n_0} \cap [T] = \{\xi\}$. *Then* $\mathrm{Aut}(P(Q_T^{\mathbb{Z}_p})) \simeq \mathbb{Z}_p^{<\omega}$.

We have now an improved analog of Theorem 8.

THEOREM 8*: *Assume* $\neg\exists n_0 \in \mathbb{N}\, \forall \xi \in [T]\; N_{\xi\restriction n_0} \cap [T] = \{\xi\}$. *Then* $\mathrm{Aut}(P(Q_T^{\mathbb{Z}_p})) \simeq \mathbb{Z}_p^{\mathbb{N}}$.

Proof: As in Theorem 8 we can prove $\mathrm{card}(\mathrm{Aut}(P(Q_T^{\mathbb{Z}_p}))) = 2^{\aleph_0}$. Since, by Lemma 14, $\mathrm{Aut}(P(Q_T^{\mathbb{Z}_p})) \simeq C_T$, each element of $\mathrm{Aut}(P(Q_T^{\mathbb{Z}_p}))$ has order p and $\mathrm{Aut}(P(Q_T^{\mathbb{Z}_p}))$ is commutative; so we deduce that $\mathrm{Aut}(P(Q_T^{\mathbb{Z}_p}))$ is a vector space on $\mathbb{Z}_p$ of dimension $2^{\aleph_0}$, thus it is isomorphic (as a vector space and so as a group) to $\mathbb{Z}_p^{\mathbb{N}}$.

LEMMA 16: $\{T \in \mathrm{Tr} \mid \neg\exists n_0 \in \mathbb{N}\, \forall \xi \in [T]\; N_{\xi\restriction n_0} \cap [T] = \{\xi\}\}$ *is complete analytic in* Tr.

Proof: First, to see that $\{T \in \mathrm{Tr} \mid \neg\exists n_0 \in \mathbb{N}\, \forall \xi \in [T]\; N_{\xi\restriction n_0} \cap [T] = \{\xi\}\}$ is analytic, we have, dealing with the complement, $\forall T \in \mathrm{Tr}(\exists n_0 \in \mathbb{N}\, \forall \xi \in \mathcal{N}\; (\xi \in [T] \Rightarrow N_{\xi\restriction n_0} \cap [T] = \{\xi\}) \iff \exists n_0 \in \mathbb{N}\, \forall \xi \in \mathcal{N}\; (\forall m \in \mathbb{N}\; \xi \restriction m \in T \Rightarrow \forall \eta \in \mathcal{N}\; (\forall k < n_0\; \eta(k) = \xi(k) \Rightarrow \forall k \in \mathbb{N}\; \eta(k) = \xi(k) \vee \exists k \in \mathbb{N}\; \eta \restriction k \notin T)))$.

To prove the hardness part we reduce IF to our set. For $T \in \mathrm{Tr}$ let T' be the tree consisting of an infinite branch $\{0^n\}_{n\in\mathbb{N}}$ with a copy of T attached to each node 0^n of that infinite branch. The assignment $T \mapsto T'$ is Borel and for

$T \in \mathrm{WF}$, T' has a unique infinite branch, while, for $T \in \mathrm{IF}$, the branch $\{0^n\}_{n\in\mathbb{N}}$ is non-isolated in T', thus T' is in the set we are interested in.

Another reduction doing the job is the function Θ described in the proof of Theorem 10.

THEOREM 11: *For any countable language $\mathcal{L}$ and prime number p we have $\{x \in X_{\mathcal{L}} \mid \mathrm{Aut}(\mathcal{A}_x) \simeq \mathbb{Z}_p^{\mathbb{N}}\} \in \mathbf{\Pi}_1^1\&\mathbf{\Sigma}_1^1(X_{\mathcal{L}})$.*

Proof: Since the automorphism group of a structure in $X_{\mathcal{L}}$ is a closed subset of S_∞, it satisfies the continuum hypothesis. So, appealing to the same argument used at the end of the proof of Theorem 8*, $\forall x \in X_{\mathcal{L}}$ $(\mathrm{Aut}(\mathcal{A}_x) \simeq \mathbb{Z}_p^{\mathbb{N}} \iff \forall g \in S_\infty\ (g \in \mathrm{Aut}(\mathcal{A}_x) \Rightarrow g^p = \mathrm{id}) \wedge \forall g \in S_\infty\ \forall h \in S_\infty\ (g \in \mathrm{Aut}(\mathcal{A}_x) \wedge h \in \mathrm{Aut}(\mathcal{A}_x) \Rightarrow gh = hg) \wedge \mathrm{card}(\mathrm{Aut}(\mathcal{A}_x)) > \aleph_0)$. Since $F = \{(x,g) \in X_{\mathcal{L}} \times S_\infty \mid g \in \mathrm{Aut}(\mathcal{A}_x)\} \in \mathbf{\Pi}_1^0(X_{\mathcal{L}} \times S_\infty)$, the first and second condition in the preceding conjunction are coanalytic, while the third is analytic.

THEOREM 12: *Let $L = \{f_n\}_{n\in\mathbb{N}}$ be a language consisting of a countable infinity of unary function symbols. Then $\{x \in X_L \mid \mathrm{Aut}(\mathcal{A}_x) \simeq \mathbb{Z}_p^{\mathbb{N}}\}$ is $\mathbf{\Pi}_1^1\&\mathbf{\Sigma}_1^1$-complete in X_L.*

Proof: Let $C = \{T \in \mathrm{Tr} \mid \neg\exists n_0 \in \mathbb{N}\ \forall \xi \in [T]\ N_{\xi\restriction n_0} \cap [T] = \{\xi\}\}$. Since C is complete analytic by lemma 16, $Z = \mathrm{WF} \times C$ is $\mathbf{\Pi}_1^1\&\mathbf{\Sigma}_1^1$-complete. So it is enough to Borel reduce it to $\{x \in X_L \mid \mathrm{Aut}(\mathcal{A}_x) \simeq \mathbb{Z}_p^{\mathbb{N}}\}$.

Let G be a non-Abelian group and let Φ_G: $\mathrm{Tr} \longrightarrow X_L$ be defined as in the proof of Theorem 4. Let Φ_p: $\mathrm{Tr} \longrightarrow X_L$ be a Borel map assigning to each $T \in \mathrm{Tr}$ a code for $P(Q_T^{\mathbb{Z}_p})$. Finally, let Ψ: $\mathrm{Tr}^2 \longrightarrow X_L$ be defined by $\mathcal{A}_{\Psi(T,V)} = \mathcal{A}_{\Phi_G(T)} \oplus \mathcal{A}_{\Phi_p(V)}$, where the direct sum $\oplus$ is defined as in the proof of Theorem 4. Again we have $\mathrm{Aut}(\mathcal{A}_{\Psi(T,V)}) \simeq \mathrm{Aut}(P(S_T^G)) \times \mathrm{Aut}(P(Q_V^{\mathbb{Z}_p}))$.

We claim that $\forall T,V \in \mathrm{Tr}(Z(T,V) \iff \mathrm{Aut}(\mathcal{A}_{\Psi(T,V)}) \simeq \mathbb{Z}_p^{\mathbb{N}})$. Indeed, if $Z(T,V)$ holds, then we have $\mathrm{Aut}(P(S_T^G)) = \{1\}$, $\mathrm{Aut}(P(Q_T^{\mathbb{Z}_p})) \simeq \mathbb{Z}_p^{\mathbb{N}}$, so $\mathrm{Aut}(\mathcal{A}_{\Psi(T,V)}) \simeq \mathbb{Z}_p^{\mathbb{N}}$. If $Z(T,V)$ fails, there are two cases: $T \notin \mathrm{WF}$ or $T \in \mathrm{WF} \wedge V \notin C$. In the first case $\mathrm{Aut}(P(S_T^G))$ contains a subgroup isomorphic to G, so $\mathrm{Aut}(\mathcal{A}_{\Psi(T,V)})$ is not Abelian, containing a subgroup isomorphic to $\mathrm{Aut}(P(S_T^G))$, and cannot be isomorphic to $\mathbb{Z}_p^{\mathbb{N}}$. In the second case $\mathrm{Aut}(\mathcal{A}_{\Psi(T,V)}) \simeq \mathrm{Aut}(P(Q_T^{\mathbb{Z}_p})) \not\simeq \mathbb{Z}_p^{\mathbb{N}}$ by Theorems 1*, 6* and 7*.

THEOREM 12′: *Let $L' = \{R\}$ be a language consisting of one binary relation symbol and let $\mathcal{G} \subseteq X_{L'}$ be the class of (codes for) graphs. Then $\{x \in \mathcal{G} \mid \mathrm{Aut}(\mathcal{A}_x) \simeq \mathbb{Z}_p^{\mathbb{N}}\}$ and $\{x \in X_{L'} \mid \mathrm{Aut}(\mathcal{A}_x) \simeq \mathbb{Z}_p^{\mathbb{N}}\}$ are $\mathbf{\Pi}_1^1\&\mathbf{\Sigma}_1^1$-complete.*

Proof: By Theorem 12 and the reduction discussed after Theorem 10.

Definition: For $n \in \mathbb{N}$ let $B_n = \{T \in \mathrm{Tr} \mid \mathrm{card}([T]) = n\}$ (so, in particular, $B_0 = \mathrm{WF}$ and $B_1 = \mathrm{UB}$). Let also $B_\omega = \{T \in \mathrm{Tr} \mid \mathrm{card}([T]) = \aleph_0 \wedge \exists n_0 \in \mathbb{N}\ \forall \xi \in [T]\ N_{\xi \restriction n_0} \cap [T] = \{\xi\}\}$ and $B_\infty = \{T \in \mathrm{Tr} \mid \neg \exists n_0 \in \mathbb{N}\ \forall \xi \in [T]\ N_{\xi \restriction n_0} \cap [T] = \{\xi\}\}$.

LEMMA 17: *For $\lambda \in \omega + 1$, B_λ and B_∞ are Borel inseparable in* Tr.

Proof: In the remark at the very end of the proof of Theorem 10, we observed the Borel inseparability of WF and $\{T \in \mathrm{Tr} \mid \mathrm{card}([T]) = 2^{\aleph_0}\}$, which implies the Borel inseparability of WF and B_∞. Applying a Borel function $\mathrm{Tr} \longrightarrow \mathrm{Tr}$ which adds to each tree λ infinite branches isolated at the root, we get the desired result.

For the case of the group $\mathbb{Z}_p$ we can thus extend a bit the inseparability results of Theorems 4, 5, 4′ and 5′ as follows.

THEOREM 13: *Let $L = \{f_n\}_{n \in \mathbb{N}}$ be a language consisting of a countable infinity of unary function symbols and put $\forall m \in \mathbb{N}\ A_m = \{x \in X_L \mid \mathrm{Aut}(\mathcal{A}_x) \simeq \mathbb{Z}_p^m\}$, $A_\omega = \{x \in X_L \mid \mathrm{Aut}(\mathcal{A}_x) \simeq \mathbb{Z}_p^{<\omega}\}$, $A_\infty = \{x \in X_L \mid \mathrm{Aut}(\mathcal{A}_x) \simeq \mathbb{Z}_p^{\mathbb{N}}\}$. Then for $\lambda \in \omega + 1$, A_λ and A_∞ are Borel inseparable.*

THEOREM 14: *Let $L = \{f_n\}_{n \in \mathbb{N}}$ be a language consisting of a countable infinity of unary function symbols. Fix $\lambda \in \omega + 1$ and let σ be an $L_{\omega_1\omega}$-sentence satisfied by all countably infinite structures whose group of automorphisms is isomorphic to $\mathbb{Z}_p^\lambda$ if $\lambda \in \mathbb{N}$ and to $\mathbb{Z}_p^{<\omega}$ if $\lambda = \omega$. Then there is a countably infinite structure for L whose automorphism group is isomorphic to $\mathbb{Z}_p^{\mathbb{N}}$ satisfying σ (in addition to a countable infinite structure for every abstract countable group, as by theorem 5).*

THEOREM 13′: *Let $L' = \{R\}$ be a language consisting of one binary relation symbol and let $\mathcal{G} \subseteq X_{L'}$ be the class of (codes for) countably infinite graphs. Put $\forall m \in \mathbb{N}\ A_1^m = \{x \in X_{L'} \mid \mathrm{Aut}(\mathcal{A}_x) \simeq \mathbb{Z}_p^m\}$, $A_1^\omega = \{x \in X_{L'} \mid \mathrm{Aut}(\mathcal{A}_x) \simeq \mathbb{Z}_p^{<\omega}\}$, $A_1^\infty = \{x \in X_{L'} \mid \mathrm{Aut}(\mathcal{A}_x) \simeq \mathbb{Z}_p^{\mathbb{N}}\}$ and $\forall \lambda \in \omega + 1 \cup \{\infty\}\ A_2^\lambda = A_1^\lambda \cap \mathcal{G}$. Then for $j \in \{1, 2\}$ and $\lambda \in \omega + 1$, A_j^λ and A_j^∞ are Borel inseparable.*

THEOREM 14′: *Let $L' = \{R\}$ be a language consisting of one binary relation symbol. Let $\lambda \in \omega + 1$ and let σ be an $L'_{\omega_1\omega}$-sentence satisfied by all countably infinite graphs whose automorphism group is isomorphic to $\mathbb{Z}_p^\lambda$ if $\lambda \in \mathbb{N}$ and to $\mathbb{Z}_p^{<\omega}$ if $\lambda = \omega$. Then there is a countably infinite graph whose automorphism group is isomorphic to $\mathbb{Z}_p^{\mathbb{N}}$ satisfying σ.*

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